CSE565M: Acceleration of Algorithms in Reconfigurable Logic

Learn by Doing: Fast Fourier Transforms (Pt. 1)

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Where We're Headed

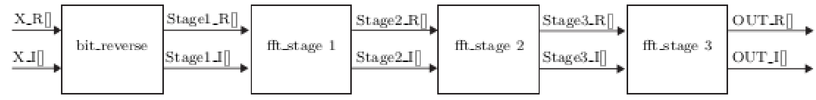


Figure 1: Dividing the FFT into different stages allows for task pipelining across each of these stages. The figure shows an example with three FFT stages (i.e., an 8 point FFT).

FFT Background

A More Efficient DFT

- Performing the DFT directly using matrix-vector multiply requires $\mathcal{O}(n^2)$ multiply and add operations, for an input signal with n samples.
- It is possible to reduce the complexity by exploiting the structure of the constant coefficients in the matrix.
- This S matrix encodes the coefficients of the DFT; each row of this matrix corresponds to a fixed number of rotations around the complex unit circle.
- These values have a significant amount of redundancy, and that can be exploited to reduce the complexity of the algorithm.

Foundation Building

- The FFT brings about a reduction in complexity by taking advantage of symmetries in the DFT.
- Recall that the DFT performs a matrix vector multiplication, i.e., $G[] = S[][] \cdot g[]$, where $g[]$ is the input data, $G[]$ is the frequency domain output data, and $S[][]$ are the DFT coefficients.

2 Point FFT

Foundation Building: 2 Point FFT

For a 2 point DFT, the values of S are:

$$S = \begin{bmatrix} W_2^{00} & W_2^{01} \\ W_2^{10} & W_2^{11} \end{bmatrix} \tag{1}$$

Here we use the notation $W = e^{-j2\pi}$.

- The $e^{-j2\pi}$ or W terms are often called **twiddle factors**.
- The superscript on W denotes values that are added to the numerator and the subscript on the W indicates those values added in the denominator of the complex exponential.
- For example, $W_4^{23} = e^{\frac{-j2\pi \cdot 2 \cdot 3}{4}}$.
- This is similar to the s value used in the DFT discussion where $s = e^{\frac{-j2\pi}{N}}$. The relationship between s and W is $s = W_N$.

Calculating the 2 Point FFT

Recalling $W_Z^{XY} = e^{\frac{-j2\pi \cdot X \cdot Y}{Z}}$.

$$\begin{bmatrix} G[0] \\ G[1] \end{bmatrix} = \begin{bmatrix} W_2^{00} & W_2^{01} \\ W_2^{10} & W_2^{11} \end{bmatrix} \cdot \begin{bmatrix} g[0] \\ g[1] \end{bmatrix} \quad (2)$$

Expanding the two equations for a 2 point DFT gives us:

$$\begin{aligned} G[0] &= g[0] \cdot e^{\frac{-j2\pi \cdot 0 \cdot 0}{2}} + g[1] \cdot e^{\frac{-j2\pi \cdot 0 \cdot 1}{2}} \\ &= g[0] + g[1] \end{aligned} \quad (3)$$

due to the fact that since $e^0 = 1$. The second frequency term

$$\begin{aligned} G[1] &= g[0] \cdot e^{\frac{-j2\pi \cdot 1 \cdot 0}{2}} + g[1] \cdot e^{\frac{-j2\pi \cdot 1 \cdot 1}{2}} \\ &= g[0] - g[1] \end{aligned} \quad (4)$$

since $e^{\frac{-j2\pi \cdot 1 \cdot 1}{2}} = e^{-j\pi} = -1$.

Visualizing the 2 Point DFT

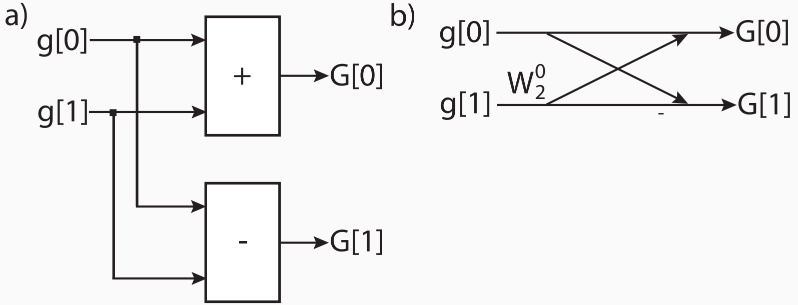
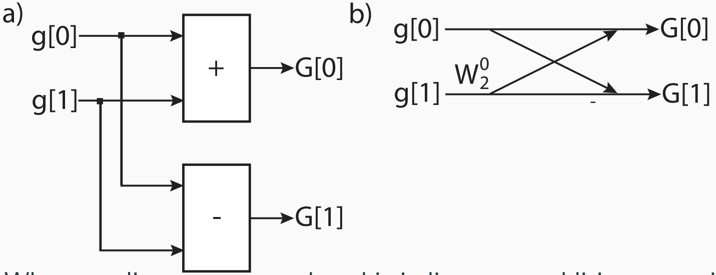


Figure 2: Part a) is a **dataflow graph** for a 2 point DFT/FFT. Part b) shows the same computation, but viewed as a **butterfly structure**. This is a common representation for the computation of an FFT in the digital signal processing domain.



- When two lines come together this indicates an addition operation.
- Any label on the line itself indicates a multiplication of that label by the value on that line.
- There are two labels in this figure.
  - The '−' sign on the bottom horizontal line indicates that this value should be negated.
  - This followed by the addition denoted by the two lines intersecting is the same as subtraction. The second label is W_2^0.

4 Point FFT

Computing the 4 Point FFT

Now let us consider a slightly larger dft – a 4 point dft, i.e., one that has 4 inputs, 4 outputs, and a 4×4 S matrix. The values of S for a 4 point dft are:

$$S = \begin{bmatrix} W_4^{00} & W_4^{01} & W_4^{02} & W_4^{03} \\ W_4^{10} & W_4^{11} & W_4^{12} & W_4^{13} \\ W_4^{20} & W_4^{21} & W_4^{22} & W_4^{23} \\ W_4^{30} & W_4^{31} & W_4^{32} & W_4^{33} \end{bmatrix} \tag{5}$$

And the dft equation to compute the frequency output terms are:

$$\begin{bmatrix} G[0] \\ G[1] \\ G[2] \\ G[3] \end{bmatrix} = \begin{bmatrix} W_4^{00} & W_4^{01} & W_4^{02} & W_4^{03} \\ W_4^{10} & W_4^{11} & W_4^{12} & W_4^{13} \\ W_4^{20} & W_4^{21} & W_4^{22} & W_4^{23} \\ W_4^{30} & W_4^{31} & W_4^{32} & W_4^{33} \end{bmatrix} \cdot \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ g[3] \end{bmatrix} \tag{6}$$

Computing the 4 Point FFT

Now we write out the equations for each of the frequency domain values in $G[]$ one-by-one. The equation for G[0] is:

$$
\begin{aligned}
G[0] &= g[0] \cdot e^{\frac{-j2\pi \cdot 0 \cdot 0}{4}} + g[1] \cdot e^{\frac{-j2\pi \cdot 0 \cdot 1}{4}} + g[2] \cdot e^{\frac{-j2\pi \cdot 0 \cdot 2}{4}} + g[3] \cdot e^{\frac{-j2\pi \cdot 0 \cdot 3}{4}} \\
&= g[0] + g[1] + g[2] + g[3]
\end{aligned} \tag{7}
$$

since $e^0 = 1$.

The equation for $G[1]$ is:

$$
\begin{aligned}
G[1] &= g[0] \cdot e^{\frac{-j2\pi \cdot 1 \cdot 0}{4}} + g[1] \cdot e^{\frac{-j2\pi \cdot 1 \cdot 1}{4}} + g[2] \cdot e^{\frac{-j2\pi \cdot 1 \cdot 2}{4}} + g[3] \cdot e^{\frac{-j2\pi \cdot 1 \cdot 3}{4}} \\
&= g[0] + g[1] \cdot e^{\frac{-j2\pi}{4}} + g[2] \cdot e^{\frac{-j4\pi}{4}} + g[3] \cdot e^{\frac{-j6\pi}{4}} \\
&= g[0] + g[1] \cdot e^{\frac{-j2\pi}{4}} + g[2] \cdot e^{-j\pi} + g[3] \cdot e^{\frac{-j2\pi}{4}} e^{-j\pi} \\
&= g[0] + g[1] \cdot e^{\frac{-j2\pi}{4}} - g[2] - g[3] \cdot e^{\frac{-j2\pi}{4}}
\end{aligned} \tag{8}
$$

The reductions were done based upon the fact that $e^{-j\pi} = -1$.

Computing the 4 Point FFT

The equation for $G[2]$ is:

$$\begin{aligned} G[2] &= g[0] \cdot e^{\frac{-j2\pi \cdot 2 \cdot 0}{4}} + g[1] \cdot e^{\frac{-j2\pi \cdot 2 \cdot 1}{4}} + g[2] \cdot e^{\frac{-j2\pi \cdot 2 \cdot 2}{4}} + g[3] \cdot e^{\frac{-j2\pi \cdot 2 \cdot 3}{4}} \\ &= g[0] + g[1] \cdot e^{\frac{-j4\pi}{4}} + g[2] \cdot e^{\frac{-j8\pi}{4}} + g[3] \cdot e^{\frac{-j12\pi}{4}} \\ &= g[0] - g[1] + g[2] - g[3] \end{aligned} \tag{9}$$

The reductions were done by simplifications based upon rotations. E.g., $e^{\frac{-j8\pi}{4}} = 1$ and $e^{\frac{-12j\pi}{4}} = -1$ since in both cases use the fact that $e^{-j2\pi}$ is equal to 1. In other words, any complex exponential with a rotation by 2π is equal.

Finally, the equation for $G[3]$ is:

$$\begin{aligned} G[3] &= g[0] \cdot e^{\frac{-j2\pi \cdot 3 \cdot 0}{4}} + g[1] \cdot e^{\frac{-j2\pi \cdot 3 \cdot 1}{4}} + g[2] \cdot e^{\frac{-j2\pi \cdot 3 \cdot 2}{4}} + g[3] \cdot e^{\frac{-j2\pi \cdot 3 \cdot 3}{4}} \\ &= g[0] + g[1] \cdot e^{\frac{-j6\pi}{4}} + g[2] \cdot e^{\frac{-j12\pi}{4}} + g[3] \cdot e^{\frac{-j18\pi}{4}} \\ &= g[0] + g[1] \cdot e^{\frac{-j6\pi}{4}} - g[2] + g[3] \cdot e^{\frac{-j10\pi}{4}} \\ &= g[0] + g[1] \cdot e^{\frac{-j6\pi}{4}} - g[2] - g[3] \cdot e^{\frac{-j6\pi}{4}} \end{aligned} \tag{10}$$

...t of the reductions that we have not seen yet deal with the last term

Reducing $e^{\frac{-j18\pi}{4}}$ to $-e^{\frac{-j6\pi}{4}}$

- It is reduced to $e^{\frac{-j10\pi}{4}}$ since these are equivalent based upon a 2π rotation, or, equivalently, $e^{\frac{-j10\pi}{4}} \cdot e^{\frac{-j8\pi}{4}}$ and the second term $e^{\frac{-j8\pi}{4}} = 1$.
- Finally, a rotation of π, which is equal to -1, brings it to $e^{\frac{-j6\pi}{4}}$. Another way of viewing this is $e^{\frac{-j6\pi}{4}} \cdot e^{\frac{-j4\pi}{4}}$ and $e^{\frac{-j4\pi}{4}} = -1$. We leave this term in this unreduced state in order to demonstrate symmetries in the following equations on the next slide.

Computing the 4 Point FFT

With a bit of reordering, we can view these four equations as:

$$\begin{aligned}
G[0] &= (g[0] + g[2]) + e^{\frac{-j2\pi 0}{4}}(g[1] + g[3]) \\
G[1] &= (g[0] - g[2]) + e^{\frac{-j2\pi 1}{4}}(g[1] - g[3]) \\
G[2] &= (g[0] + g[2]) + e^{\frac{-j2\pi 2}{4}}(g[1] + g[3]) \\
G[3] &= (g[0] - g[2]) + e^{\frac{-j2\pi 3}{4}}(g[1] - g[3])
\end{aligned} \tag{11}$$

Exploitable symmetries:

- partition input data nto even and odd elements, i.e., similar operations for $g[0]$ and $g[2]$, and $g[1]$ and $g[3]$.
- addition and subtraction symmetries on these even and odd elements, e.g., for $G[0]$ and $G[2]$, the even and odd elements are summed, and hare subtracted when calculating the frequencies $G[1]$ and $G[3]$.
- the odd elements in every frequency term are multiplied by a constant complex exponential W_4^i where i denotes the index for the frequency output, i.e., $G[i]$.

Computing the 4 Point FFT

$$
\begin{aligned}
G[0] &= (g[0] + g[2]) + e^{\frac{-j2\pi 0}{4}}(g[1] + g[3]) \\
G[1] &= (g[0] - g[2]) + e^{\frac{-j2\pi 1}{4}}(g[1] - g[3]) \\
G[2] &= (g[0] + g[2]) + e^{\frac{-j2\pi 2}{4}}(g[1] + g[3]) \\
G[3] &= (g[0] - g[2]) + e^{\frac{-j2\pi 3}{4}}(g[1] - g[3])
\end{aligned}
$$

Looking at the terms in the parentheses, we see that they are the same as 2 point FFTs!

- consider the terms corresponding to the even input values $g[0]$ and $g[2]$. If we perform a 2 point FFT on these even terms, the lower frequency (DC value) is $g[0] + g[2]$, and the higher frequency is calculated as $g[0] - g[2]$.
- The same is true for the odd input values $g[1]$ and $g[3]$.

4 Point FFT

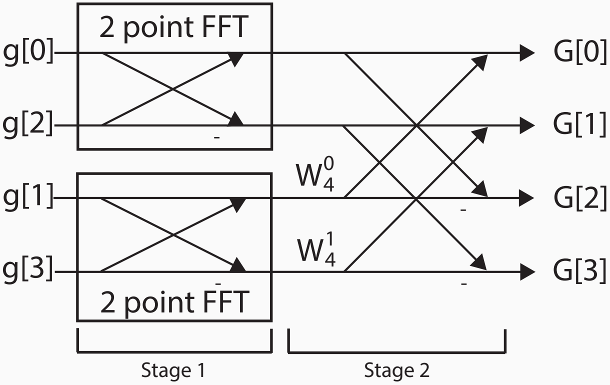


Figure 3: A four point FFT divided into two stages. Stage 1 has uses two 2 point FFTs – one 2 point FFT for the even input values and the other 2 point FFT for the odd input values. Stage 2 performs the remaining operations to complete the FFT computation.

Computational Complexity Reduction

We are now starting to build intuition for the reduction in complexity from $\mathcal{O}(n^2)$ operations for the dft to $\mathcal{O}(n \log n)$ operations for the FFT. The key idea is building the computation through recursion.

- The 4 point FFT uses two 2 point FFTs. This extends to larger FFT sizes. For example, an 8 point FFT uses two 4 point FFTs, which in turn each use two 2 point FFTs (for a total of four 2 point FFTs). An 16 point FFT uses two 8 point FFTs, and so on.

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Generalizing the number of 2 Point FFTs needed

How many 2 point FFTs are used in a 32 point FFT? How many are there in a 64 point FFT? How many 4 point FFTs are required for a 64 point FFT? How about a 128 point FFT? What is the general formula for 2 point, 4 point, and 8 point FFTs in an N point FFT (where $N > 8$)?

Generalizing to the N Point FFT

Deriving the Recursive Relationship

Assume that we are calculating an N point FFT. The formula for calculating the frequency domain values $G[]$ given the input values $g[]$ is:

$$G[k] = \sum_{n=0}^{N-1} g[n] \cdot e^{\frac{-j2\pi kn}{N}} \text{ for } k = 0, \ldots, N-1 \tag{12}$$

We can divide this equation into two parts, one that sums the even components and one that sums the odd components.

$$G[k] = \sum_{n=0}^{N/2-1} g[2n] \cdot e^{\frac{-j2\pi k(2n)}{N}} + \sum_{n=0}^{N/2-1} g[2n+1] \cdot e^{\frac{-j2\pi k(2n+1)}{N}} \tag{13}$$

Note that the sums now go to $N/2 - 1$ in both cases which should make sense since we have divided them into two halves.

Deriving the Recursive Relationship

We transform Equation 13 to the following:

$$G[k] = \sum_{n=0}^{N/2-1} g[2n] \cdot e^{\frac{-j2\pi kn}{N/2}} + \sum_{n=0}^{N/2-1} g[2n+1] \cdot e^{\frac{-j2\pi k(2n)}{N}} \cdot e^{\frac{-j2\pi k}{N}} \quad (14)$$

- 1st summation (even inputs), move 2 to the denominator for $N/2$.
- 2nd summation (odd inputs) uses the power rule to separate the $+1$ leaving two complex exponentials.

Deriving the Recursive Relationship

We can further modify Equation 14 to

$$G[k] = \sum_{n=0}^{N/2-1} g[2n] \cdot e^{\frac{-j2\pi kn}{N/2}} + e^{\frac{-j2\pi k}{N}} \cdot \sum_{n=0}^{N/2-1} g[2n+1] \cdot e^{\frac{-j2\pi kn}{N/2}} \qquad (15)$$

Modify the second summation.

- pull one of the complex exponentials outside of the summation since it does not depend upon n.
- And we also move the 2 into the denominator as we did before in the first summation.
    - Note that both summations now have the same complex exponential $e^{\frac{-j2\pi kn}{N/2}}$.

Deriving the Recursive Relationship

Finally, we simplify this to

$$G[k] = A_k + W_N^k B_k \tag{16}$$

where A_k and B_k are the first and second summations, respectively. And recall that $W = e^{-j2\pi}$. This completely describes an N point FFT by separating even and odd terms into two summations.

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For reasons that will become clear soon, let us assume that we

- only want to use Equation 16 to calculate the first $N/2$ terms, i.e., $G[0]$ through $G[N/2-1]$.
- will derive the remaining $N/2$ terms, i.e., those from $G[N/2]$ to $G[N-1]$ using a different equation.

Deriving the Recursive Relationship

In order to calculate the higher frequencies $G[N/2]$ to $G[N-1]$, let us derive the same equations but this time using $k = N/2, N/2+1, \ldots, N/2-1$. Thus, we wish to calculate

$$G[k+N/2] = \sum_{n=0}^{N-1} g[n] \cdot e^{\frac{-j2\pi(k+N/2)n}{N}} \text{ for } k = 0, \ldots, N/2-1 \tag{17}$$

Using similar algebra from the previous slides, we arrive at:

$$G[k+N/2] = \sum_{n=0}^{N/2-1} g[2n] \cdot e^{\frac{-j2\pi kn}{N/2}} - e^{\frac{-j2\pi k}{N}} \cdot \sum_{n=0}^{N/2-1} g[2n+1] \cdot e^{\frac{-j2\pi kn}{N/2}} \tag{18}$$

Note the similarity to Equation 15. We can put it in terms of Equation 16 as

$$G[k+N/2] = A_k - W_N^k B_k \tag{19}$$

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N Point FFT

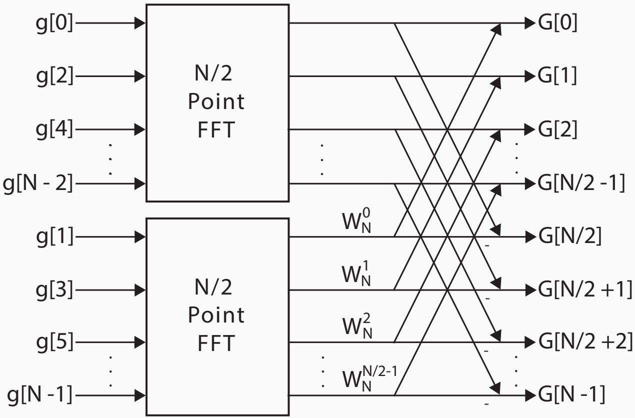
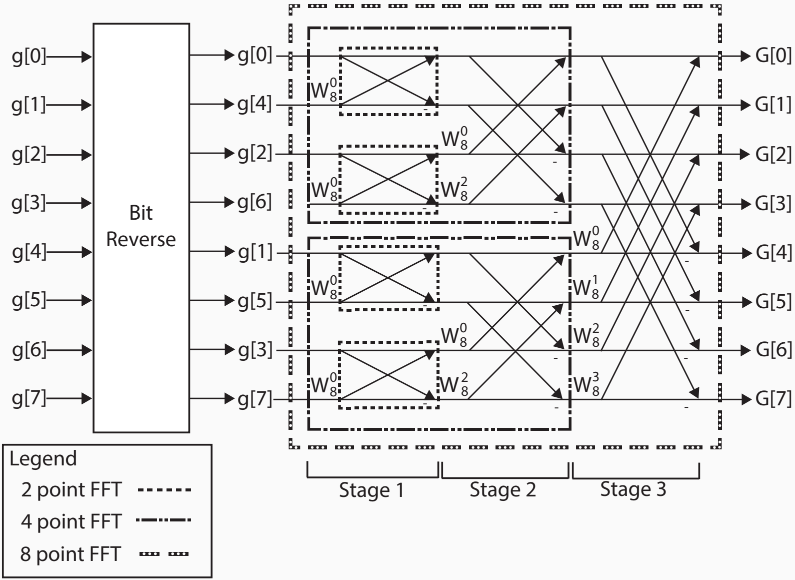


Figure 4: Building an N point FFT from two $N/2$ point FFTs. The upper $N/2$ point FFT is performed on the even inputs; the lower $N/2$ FFT uses the odd inputs.

Recursively Building an 8-pt FFT



Recursively Building an 8-pt FFT

Also note that the inputs must be reordered before they are feed into the 8 point FFT.

- This is due to the fact that the different $N/2$ point FFTs take even and odd inputs.
- The upper four inputs correspond to even inputs and the lower four inputs have odd indices. However, they are reordered twice. If we separate the even and odd inputs once we have the even set $\{g[0], g[2], g[4], g[6]\}$ and the odd set $\{g[1], g[3], g[5], g[7]\}$.

Now let us reorder the even set once again. In the even set $g[0]$ and $g[4]$ are the even elements, and $g[2]$ and $g[6]$ are the odd elements. Thus reordering it results in the set $\{g[0], g[4], g[2], g[6]\}$. The same can be done for the initial odd set yielding the reordered set $\{g[1], g[5], g[3], g[7]\}$.

The final reordering is done by swapping values whose indices are in bit reversed order.

Recursively Building FFT

Table 1: The index, three bit binary value for that index, bit reversed binary value, and the resulting bit reversed index.

Index	Binary	Reversed Binary	Reversed Index
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

Baseline Implementation and Optimizations (next time)

References i